18 December 2000

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 278 (2000) 44–46

www.elsevier.nl/locate/pla

Quantum roulette: an extended quantum strategy

Xiang-Bin Wang, L.C. Kwek, C.H. Oh *

Department of Physics, Faculty of Science, National University of Singapore, Lower Kent Ridge, Singapore 119260, Singapore

Received 17 July 2000; received in revised form 9 October 2000; accepted 20 November 2000
Communicated by A.R. Bishop

Abstract

In a recent paper, Meyer demonstrated that with a quantum computer, an analogous zero-sum classically strategic game played with quantum strategy essentially become a bias game under a mixture of quantum and classical strategy. To illustrate his point, Meyer used a quantum coin tossing event. In this Letter, we generalize Meyer's argument to an N-state game. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Quantum entanglement; Quantum games; Quantum strategy and coin tossing

---

The construction of game theoretic models has always been an interesting and relevant tool for the decision making process in industrial organization and economic theories [1,2]. Recently, the classical notion of game theory has been generalized to include quantum games [3,4] and extended to quantum gambling [5]. Indeed, quantum eavesdropping [6,7] and the optimal copying of states [8] can also be regarded as strategic games between two or more players.

In a recent paper [3], Meyer demonstrated that in a classically two-person zero-sum strategic game, if one person adopts a quantum strategy, then he has a better chance of winning the game. Meyer's strategy is as follows: two persons P and Q take turn to flipped a coin. P initially places the coin head up in a box. Thereafter, Q, then P and Q take turns to flip the coin. Q wins if the coin is head up and loses otherwise. Classically, this game is a zero-sum strategy, that is to say it is a fair game.

In the quantum version, the initial state of the coin is represented by a density matrix, ρ_0, so that in the basis $\{|H\rangle, |T\rangle\}$, where symbols H and T denote head and tail, respectively, and ρ_0 is given by

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Suppose Q adopts a quantum strategy, then Q uses a unitary rather than a stochastic matrix to act on the coin. Let this unitary transformation be U_1, so that the state of the coin at the end of the transformation is given by $\rho_1 = U_1 \rho_0 U_1^\dagger$. Player P, however, continues to play with a classical probabilistic strategy. Thus, P employs a convex sum of unitary (deterministic) transformation, namely he either flips the coin using the transformation F with probability p or lets the coin rest in its original state (using the identity transformation) with probability $(1-p)$, where

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

---

[*] Corresponding author.
E-mail addresses: phywangx@nus.edu.sg (X.-B. Wang), phyohch@nus.edu.sg (C.H. Oh).

0375-9601/00/$ – see front matter
PII: S0375-9601(00)00771-4

Thus, at the end of P's turn, the state of the coin should be described by the density matrix

$$\rho_2 = pF\rho_1F^\dagger + (1-p)\rho_1. \tag{3}$$

Finally, Q "flips" the coin using the unitary transformation, U_2, so that the final state of the coin is $\rho_3 = U_2\rho_2U_2^\dagger$. Meyer has shown that if Q selects the unitary matrices $U_1 = U_2 = H$, where H is the Hadamard transform given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{4}$$

then $\rho_3 = \rho_0$, independent of the probability p. Thus Q wins the game all the time.

Meyer's strategy is really a quantum coin-tossing event. In fact, we can generalize the situation by replacing the coin which has only two possible states (namely head and tail) with a roulette with N states. Consider the situation for $N = 3$. In this case, the generalization of the initial state reads

$$\rho_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

Using a quantum strategy with

$$U_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}, \tag{6}$$

player Q first rotates the initial density matrix ρ_0 to the form $(U_1\rho_0U_1^\dagger)_{ij} = (\rho_0')_{ij} = \frac{1}{3} \equiv M$ for $i, j = 1, \ldots, 3$. Here the matrix M is the density matrix after player Q's first flip. For this case, player P has a choice of 3! possible flips corresponding to all the possible permutations of the set $\{1, 2, 3\}$ to itself. Specifically, player P has the choice of choosing one of the following "flips" [9]:

$$\left\{ \pi_0 = 1, \quad \pi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \right.$$
$$\pi_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pi_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\left. \pi_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \pi_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}. \tag{7}$$

Notice that there is one and only one non-zero entry in each column and row. At the end of player P's turn, the density matrix takes the form

$$\begin{aligned} \rho_2 &= (1 - p_1 - \cdots - p_5)\rho_1 + p_1\pi_1\rho_1\pi_1^\dagger \\ &\quad + p_2\pi_2\rho_1\pi_2^\dagger + \cdots + p_5\pi_5\rho_1\pi_5^\dagger && \text{(8a)} \\ &= (1 - p_1 - \cdots - p_5)\rho_1 + p_1\pi_1M\pi_1^\dagger \\ &\quad + p_2\pi_2M\pi_2^\dagger + \cdots + p_5\pi_5M\pi_5^\dagger && \text{(8b)} \\ &= M. && \text{(8c)} \end{aligned}$$

Finally, player Q recovers ρ_0 by inverting M into ρ_0 using $U_2 = U_1$.

We next consider the case in which $N = 2^m$ for some integer m. Experimentally, this can be realized with a quantum roulette acting on an m-qubit. In this generalization, Q continues to adopt a quantum strategy using unitary matrices U_1 and U_2. In the generalized case, player P has a choice of $N!$ "flips" or permutations of the N states. We illustrate our generalization with $N = 4$. In this case, P can chose with probability $\{1 - p_1 - p_2 - \cdots - p_{23}, p_1, p_2, \ldots, p_{23}\}$ the permutations $\{\pi_0 = 1, \pi_1, \pi_2, \ldots, \pi_{23}\}$, where π_i denotes a permutation of the set of integers $\{1, 2, \ldots, 4\}$ to itself. Thus at the end of P's turn, the density matrix describing the state of the roulette is

$$\rho_2 = \sum_{i=0}^{23} p_i\pi_i\rho_1\pi_i, \tag{9}$$

with $\sum_i p_i = 1$. We can then show that if $U_1 = U_2 = H \otimes H$, then $\rho_3 = \rho_0$.

To do this, we note that for any permutation π_i, $\pi_iM\pi_i = M$, where the matrix M has all its entries equal to $1/N$. Note also that the matrices π_i are square matrices in which there is an entry unity in each row and each column and zero everywhere else. Since M has all entries equal, it does not matter how it is permuted.

In the case of $N = 4$, the initial density matrix can be represented as $\rho_0 \otimes \rho_0$. At the end of Q's first "flip", the density matrix describing the state of the roulette is $\rho_1 = H\rho_0H \otimes H\rho_0H = M$. This has the effect of initializing the initial state to a superposition of all other states. Using Eq. (9), one readily sees that the density matrix after P's turn assumes the form $\rho_2 = \sum_i p_i\rho_1 = \rho_1$, so that using the unitary transformation U_2, one gets $\rho_3 = \rho_0$. The result generalizes easily to any integer $N = 2^m$. Indeed, for $N \neq 2^m$, it is still possible to define an optimal strategy if one notes that it is always possible to rotate the initial density matrix under some unitary transformation to a matrix with all entries equal to $1/N$.

To summarize, we note that the idea in this generalization is to choose a unitary matrix U_1 so that the initial matrix, ρ_0 is rotated to the form M. This procedure is always possible since $\operatorname{rank}(\rho_0) = \operatorname{rank}(M) = 1$. At the end of the second player's move, the new density matrix is always M since the various permutations do not change the form of the matrix. Finally, the first player recovers the initial matrix using $U_2 = U_1$. Finally, we note that a similar extension can also be applied to the quantum prisoner's dilemma [4]. Moreover, such consideration can prove to be useful for studying the problem of communicating over a quantum Euclidean channel [10].

References

[1] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Wiley, New York, 1967.
[2] R. Gibbons, A Primer in Game Theory, Harvester Wheatsheaf, New York, 1992.
[3] D.A. Meyer, Phys. Rev. Lett. 82 (1999) 1052.
[4] J. Eisert, M. Wilkens, M. Lewenstein, Phys. Rev. Lett. 83 (1999) 3077.
[5] L. Goldenberg, L. Vaidman, S. Wiesner, Phys. Rev. Lett. 82 (1999) 3356.
[6] A.K. Ekert, Phys. Rev. Lett. 67 (1991) 661.
[7] N. Gisin, B. Huttner, Phys. Lett. A 228 (1997) 13.
[8] R.F. Werner, Phys. Rev. A 58 (1998) 1827.
[9] I.N. Herstein, Topics in Algebra, Wiley, New York, 1975.
[10] G. Hedby, IEEE Trans. Inf. Theory 40 (1994) 594.