Quantum and classical geometric phase of the time-dependent harmonic oscillator

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In a recent paper [Y. C. Ge and M. S. Child, Phys. Rev. Lett. **78**, 2507 (1997)], by using a Gaussian wave function, Ge and Child presented a nonadiabatic relation between the quantum Berry phase and the classical Hannay angle for the time-dependent harmonic oscillator. In this paper, we present a perspective for this relation without the use of a trial wave function. In particular, an exact explicit formula for the cyclic evolution over the period T in the parameter space of action invariant is obtained; the $-(n+1/2)$ relation between the quantum geometric angle and the Hannay angle is rigorously established.

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I. INTRODUCTION

Since Berry's seminal paper on the extra quantum geometric phase associated with the adiabatic evolution of a physical system [1], there has been a flurry of activities on the subject [2]. Indeed, the rich and elegant formalism of holonomy and connection lends itself naturally to the mathematical formulation of phenomenon [3]. It was observed that when the dynamical phase is removed, the evolution of the system is simply a parallel transport of the phase. Moreover, the adiabatic condition was subsequently found to be unnecessary provided the integral of the expectation of the Hamiltonian is identified as the dynamical phase [4].

The acquisition of a geometric phase in an adiabatic evolution is not confined to quantum phenomena. The classical analog exists and is sometimes referred to as the Hannay angle [5]. In another interesting paper [6,7], Berry established a semiclassical relation between the quantum and classical geometric phase in an adiabatic evolution. Specifically, Berry showed that under the adiabatic approximation, the classical Hannay angle and quantum geometric phase obey the following relation:

$$\Delta\phi_g = -\frac{\partial\gamma_n(C)}{\partial n}, \tag{1}$$

where $\Delta\theta_g$ is the adiabatic classical Hannay angle [5] and $\gamma_n(C)$ is the adiabatic geometric phase (Berry's phase). Furthermore, if the Hamiltonian is a time-dependent harmonic oscillator (TDHO) [6], the analysis can be done exactly using quantum mechanics without any semiclassical approximation. Recently, this relation between the quantum and classical geometric phase has been extended to nonadiabatic evolution for the TDHO [8–11]. To simplify computation, the authors substituted trial wave functions in their calculations in order to establish the relation

$$\beta = \langle\Phi|\frac{\partial}{\partial t}|\Phi\rangle = -\tfrac{1}{2}\,\theta_g, \tag{2}$$

where β is the quantum geometrical phase and θ_g is the nonadiabatic Hannay angle.

The purpose of this paper is to derive the same nonadiabatic relation more rigorously from a perspective using exponential quadratic operators [12]. Our derivation has the advantage that there is no assumption of a specific form for the initial wave function. The idea of gauge transformation provides an elegant tool for nonadiabatic evolution. We will show that by the same gauge transformation both the classical and quantum nonadiabatic Hamiltonian have the same diagonalized form as that of the adiabatic Hamiltonian for the TDHO. Thus, in our formalism, by invoking gauge transformation, Hannay's angle can be calculated in an analogous manner to that for Berry's phase. In Sec. II, we briefly sketch our technique. We see that the adiabatic formula in Eq. (1) is also valid for the nonadiabatic situation. In Sec. III, we establish the nonadiabatic relation between the quantum and classical geometric phase. Naturally, our result reproduces the result found in Ref. [8] when some winding numbers are set to zero.

II. METHOD

Consider the Hamiltonian

$$\hat{H} = \tfrac{1}{2}\,[a(t)\hat{x}^2 + b(t)\hat{p}^2 + c(t)(\hat{x}\hat{p} + \hat{p}\hat{x})]. \tag{3}$$

This TDHO can be solved exactly through the Lewis method [13] or through gauge transformation [14]. In our approach, we choose the latter and invoke gauge transformation [14] to solve the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{4}$$

To this end, we suppose $|\psi(t)\rangle = U|\psi'(t)\rangle$ so that the above equation can be reexpressed in terms of $|\psi'(t)\rangle$ as

$$i\frac{\partial}{\partial t}|\psi'(t)\rangle = \hat{H}'|\psi'(t)\rangle \tag{5}$$

and

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$$\hat{H}'=U^{-1}\hat{H}U-iU^{-1}\frac{\partial U}{\partial t}. \tag{6}$$

From Ref. [14], we know that the solution of Eq. (4) can be obtained through gauge transformation and can be written as

$$|\psi_n(t)\rangle=U_1(t)U_2(t)U_3(t)|n(t)\rangle, \tag{7}$$

where

$$U_1(t)=\exp\left[-\frac{i}{2}\int_0^t c(t')dt'(\hat{x}\hat{p}+\hat{p}\hat{x})\right], \tag{8}$$

$$U_2(t)=\exp\left[\frac{im(t)}{4}\left(\frac{\dot{f}}{f}+\frac{\dot{f}^*}{f^*}\right)\hat{x}^2\right], \tag{9}$$

$$U_3(t)=\exp\left[-\frac{i}{4}\ln|f|^2(\hat{x}\hat{p}+\hat{p}\hat{x})\right]. \tag{10}$$

Here, the state

$$|n(t)\rangle=\exp\left[-i\left(n+\frac{1}{2}\right)W\int_0^t \tfrac{1}{2}\,v(t')dt'\right]|n\rangle$$

is the eigenstate of the simple harmonic oscillator $H_0=\frac{1}{2}(W^2\hat{x}^2+\hat{p}^2)$, W is a real constant defined by $W=\frac{1}{2}i(ff^*-ff^*)$, and $v(t)\equiv 1/[m(t)|f(t)|^2]$. The functions $f(t)$ and $f^*(t)$ are two linearly independent solutions of equation $\ddot{q}+(m/m)\dot{q}+(k/m)q=0$, with $m(t)=\exp[2\int_0^t c(t')dt']b^{-1}(t)$ and $k(t)=\exp[2\int_0^t c(t')dt']a(t)$. Moreover, applying the Baker-Campbell-Hausdorff formula [12], the matrices U_i $(i=1,...,3)$ obey the property

$$U_i^{-1}(t)\begin{pmatrix} x\\ \partial\end{pmatrix}U_i(t)=M_i(t)\begin{pmatrix} x\\ \partial\end{pmatrix}, \tag{11}$$

where

$$M_1(t)=\begin{pmatrix}\exp\left(\int_0^t c(t')dt'\right) & 0\\ 0 & \exp\left(-\int_0^t c(t')dt'\right)\end{pmatrix}, \tag{12}$$

$$M_2(t)=\begin{pmatrix}1 & 0\\ i\,\frac{m(t)}{2}\left(\frac{\dot{f}}{f}+\frac{\dot{f}^*}{f^*}\right) & 1\end{pmatrix}, \tag{13}$$

$$M_3(t)=\begin{pmatrix}|f(t)| & 0\\ 0 & \frac{1}{|f(t)|}\end{pmatrix}. \tag{14}$$

Note that the corresponding transformation for $(\hat{x},\hat{p})$ is given by $U_k^{-1}(t,0)(\hat{x},\hat{p})U_k(t,0)=(\hat{x},\hat{p})L_k$, and

$$L_k=\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}M^T{}_k\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix},\quad k=1,2,3, \tag{15}$$

where the superscript T denotes transpose.

Finally, it is worth noting that the invariant of the time-dependent Hamiltonian is

$$\hat{I}=\tfrac{1}{2}[A(t)\hat{x}^2+B(t)\hat{p}^2+C(t)(\hat{x}\hat{p}+\hat{p}\hat{x})], \tag{16}$$

where $A(t)=\delta(t)|f|^2$, $B(t)=\delta(t)[|f|^2/b^2(t)]$, and $C(t)=-\delta(t)[\mathrm{Re}(ff^*)/b(t)]$. The function $\delta(t)$ is defined by $\delta(t)=\int_0^t c(t')dt'$. (Note that a possible application of such an action-angle variable in quantum theory can be found in Ref. [15].) A direct calculation shows that

$$\hat{I}=\tfrac{1}{2}(\hat{x},\hat{p})L_1^{-1}L_2^{-1}L_3^{-1}\begin{pmatrix}W^2 & 0\\ 0 & 1\end{pmatrix}(L_3^{-1})^T(L_2^{-1})^T(L_1^{-1})^T\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}, \tag{17}$$

$$=U_1(t)U_2(t)U_3(t)[\tfrac{1}{2}(\hat{p}^2+W^2\hat{x}^2)]U_3(t)^{-1}U_2(t)^{-1}U_1(t)^{-1}, \tag{18}$$

where L_k is given by Eq. (15). It is instructive to note that $\hat{H}'=v(t)\hat{I}$.

III. QUANTUM-CLASSICAL RELATIONSHIP

We now study the $-(n+\frac{1}{2})$ relation between the quantum geometric phase and Hannay's angle. It has been pointed out in Ref. [8] that their results ''indicate that we may treat Lewis' invariant $\hat{I}$ as an adiabatic Hamiltonian, the Berry phase and Hannay angle of which are equal to the nonadiabatic geometrical phase and angle of Hamiltonian'' in Eq. (3). We now provide further insight into this idea and establish a stricter mathematical proof of the $-(n+\frac{1}{2})$ relation for the TDHO. Let us first recall the adiabatic treatment.

Quantum mechanically, if the Hamiltonian changes very slowly with time, we can solve the Schrödinger equation approximately using the stationary eigenstate of the instantaneous Hamiltonian, i.e., we solve the equation of

$$\hat{H}(R_i(t))|n(R_i(t))\rangle=E_n|n(R_i(t))\rangle. \tag{19}$$

This state $|n(R_i(t))\rangle$ is the phase-corrected eigenstate of the Hamiltonian $\hat{H}$ under the *adiabatic* approximation. The explicit expression for $|n(R_i(t))\rangle$ can be obtained upon diagonalization of $\hat{H}$ by the Bogoliubov transformation, i.e., if the eigenstate of $\hat{H}_0$ is $|n_0(R_i(t))\rangle$, then the eigenstate of $\hat{H}$ is $U|n_0(R_i(t))\rangle$ provided that $U\hat{H}U^{-1}=\hat{H}_0$. The eigenstate obtained through adiabatic treatment is generally different from the exact (nonadiabatic) treatment. For the exact (nonadiabatic) situation of the Hamiltonian defined by Eq. (3), one can make use of gauge transformation [14]. In Ref. [14], the phase-corrected solution for TDHO is

$$|\chi_n\rangle=U_1U_2U_3|n\rangle. \tag{20}$$

As defined earlier, $|n\rangle$ is the eigenstate of the simple oscillator H_0. Moreover, the *adiabatic* solution of $\hat{H}'=v(t)\hat{I}$ can be obtained by solving the instantaneous stationary eigenstate equation in Eq. (19). We also see that the phase-corrected state $|\chi_n\rangle$, within a time-dependent phase factor, is not only an *adiabatic* solution of Hamiltonian $\hat{H}'$, but is also a nonadiabatic solution of the TDHO. By the first part of Eq. (2), the same phase-corrected state should have the same geometric phase. Thus, we conclude that *the exact geometric phase of a TDHO*, $\hat{H}$, *is equal to the adiabatic geometric phase of* $\hat{H}'$.

Classically, we can consider the same Hamiltonian defined by Eq. (3). Denoting $\xi=\binom{x}{p}$, we rewrite the Hamil *tonian as* $H=\frac{1}{2}\,\xi^T N\xi$, *and* $N=\begin{pmatrix} a(t) & c(t) \\ c(t) & b(t)\end{pmatrix}$. *The equation of motion is*

$$\dot{\xi}=JN\xi \tag{21}$$

and $J=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$. There exists a linear canonical transformation defined by $\xi=D\xi'$ such that D is a symplectic matrix satisfying $D^TJD=J$. Indeed, we have an equivalent equation of motion on ξ' since $\partial(D\xi')/\partial t=JND\xi'$ and Eq. (21) reduces to a neater form given by

$$\dot{\xi}'=JN'\xi' \tag{22}$$

with

$$N'=D^TND+JD^{-1}\frac{\partial D}{\partial t}. \tag{23}$$

This is the analog of the gauge transformation for the TDHO in quantum mechanics. Taking the transformation

$$D=L_1(t)L_2(t)L_3(t), \tag{24}$$

we get

$$\dot{\xi}'=Jv(t)\begin{pmatrix} W^2 & 0 \\ 0 & 1\end{pmatrix}\xi'. \tag{25}$$

Moreover, it can be shown that classical invariant I satisfies

$$\frac{\partial I}{\partial t}+\{I,H\}_\xi=0, \tag{26}$$

with the explicit form defined as in Eq. (17) by

$$I=\frac{1}{2}\,(x,p)D^{-1}\begin{pmatrix} W^2 & 0 \\ 0 & 1\end{pmatrix}(D^{-1})^T\begin{pmatrix} x \\ p\end{pmatrix}. \tag{27}$$

However, for the Hamiltonian

$$H'=\frac{I}{m(t)|f(t)|^2}\equiv v(t)I,$$

the equation of motion is $\dot{\xi}=Jv(t)\begin{pmatrix} A(t) & C(t) \\ C(t) & B(t)\end{pmatrix}\xi$. Denoting $\xi=D^T\xi'$, we see that an adiabatic solution of the equation of motion is

$$\dot{\xi}'=JD\begin{pmatrix} A(t) & C(t) \\ C(t) & B(t)\end{pmatrix}D^T\xi'=Jv(t)\begin{pmatrix} W^2 & 0 \\ 0 & 1\end{pmatrix}\xi'. \tag{28}$$

Comparing Eq. (25) and Eq. (28), we see that the exact solution of the equation of motion of H is equivalent to the *adiabatic* solution of the equation of motion of H'. Thus in phase space, they represent two identical tori with the corresponding points of each torus having the same time evolution. Consequently, corresponding points in the two tori will certainly enclose the same area A. Moreover, the invariant of the *adiabatic* Hamiltonian H', given by $H'/v(t)W=I$, has the same form as the invariant of the *nonadiabatic* Hamiltonian H. By the formula in Ref. [7], namely, the Hannay angle $\Delta\theta_g=-\partial\langle A\rangle/\partial I$, this implies that the nonadiabatic H and the adiabatic H' possess the same Hannay angle. Adiabatically, the Berry phase γ_n from Hamiltonian $\hat{H}'$ is related to the Hannay angle $\Delta\theta_g$ from Hamiltonian H' by Eq. (2). Therefore, *the same relation still holds for the nonadiabatic situation*. Putting Eq. (38) of Ref. [14] into Eq. (1), we immediately get the following *nonadiabatic* relationship for Hamiltonian H:

$$\gamma_n=-(n+\tfrac{1}{2})\Delta\theta_g, \tag{29}$$

which agrees with the results in Ref. [9]. Moreover, by setting $n=0$, we have $\gamma_0=-\frac{1}{2}\Delta\theta_g$, reproducing the result in Ref. [8].

IV. CONCLUSION

In summary, we reiterate the main points. We have established more rigorously Eq. (1) relating the quantum geometric phase and the classical Hannay angle using gauge transformation for the TDHO. No semiclassical assumption is invoked in our method. Moreover, the quantum and classical systems are treated in the same manner with analogous equations. Our approach should also be applicable to the driven TDHO as investigated in a recent work by Song [16].

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